

Supersymmetric Non-singular Fractional D2-branes and NS-NS 2-branes

M. Cvetič[†], G.W. Gibbons[‡], H. Lü^{*} and C.N. Pope[‡]

[†]*Department of Physics and Astronomy*

University of Pennsylvania, Philadelphia, PA 19104

[‡]*DAMTP, Centre for Mathematical Sciences, Cambridge University, Wilberforce Road,
Cambridge CB3 0WA, UK*

^{*}*Michigan Center for Theoretical Physics*

University of Michigan, Ann Arbor, Michigan 48109

[‡]*Center for Theoretical Physics*

Texas A&M University, College Station, TX 77843

^{†,‡}*Institut Henri Poincaré*

11 rue Pierre et Marie Curie, F 75231 Paris Cedex 05

ABSTRACT

We obtain regular deformed D2-brane solutions with fractional D2-branes arising as wrapped D4-branes. The space transverse to the D2-brane is a complete Ricci-flat 7-manifold of G_2 holonomy, which is asymptotically conical with principal orbits that are topologically \mathbb{CP}^3 or the flag manifold $SU(3)/(U(1) \times U(1))$. We obtain the solution by first constructing an L^2 normalisable harmonic 3-form. We also review a previously-obtained regular deformed D2-brane whose transverse space is a different 7-manifold of G_2 holonomy, with principal orbits that are topologically $S^3 \times S^3$. This describes D2-branes with fractional NS-NS 2-branes coming from the wrapping of 5-branes, which is supported by a non-normalisable harmonic 3-form on the 7-manifold. We prove that both types of solutions are supersymmetric, preserving 1/16 of the maximal supersymmetry and hence that they are dual to $\mathcal{N} = 1$ three-dimensional gauge theories. In each case, the spectrum for minimally-coupled scalars is discrete, indicating confinement in the infrared region of the dual gauge theories. We examine resolutions of other branes, and obtain necessary conditions for their regularity. The resolution of many of these seems to lie beyond supergravity. In the process of studying these questions, we construct new explicit examples of complete Ricci-flat metrics.

1 Introduction

In order to make use of the AdS/CFT correspondence [1, 2, 3] to study four-dimensional Yang-Mills theories with less than maximal supersymmetry, or with no superconformal symmetry, an extensive programme of studying D3-branes in conifold-type backgrounds has been undertaken. In the original study [4], the flat six-dimensional transverse space of the usual D3-brane was replaced by the conifold metric [5], which is the Ricci-flat cone over the 5-dimensional space $T^{1,1}$ (a $U(1)$ bundle over $S^2 \times S^2$). Additionally, since the $T^{1,1}$ space (which is topologically $S^2 \times S^3$) has a non-trivial 2'nd cohomology, it was possible to wrap D5-branes around the S^2 cycle, giving rise to supergravity duals of the so-called “fractional D3-branes” [6, 7, 8, 9]. At the level of the supergravity field theory, the wrapping of the D5-branes corresponds to having a non-trivial magnetic flux integral for the R-R 3-form field strength; in other words the field strength is proportional to the volume form of the 3-sphere. In fact both the R-R and NS-NS 3-forms are non-vanishing in the solution, taking their values from the real and imaginary parts of a self-dual harmonic 3-form. This leads to a non-trivial contribution in the Bianchi identity of the self-dual 5-form that carries the original D3-brane charge.

The conifold metric is singular at the origin $r = 0$ (*i.e.* at the apex of the cone) and furthermore, the fractional D3-brane solution based on the conifold has a naked singularity at some positive value of r [4]. Both these problems can be eliminated if one replaces the conifold by a deformation to a complete Ricci-flat manifold [10]. The six-manifold is T^*S^3 (the cotangent bundle of the 3-sphere). Its Ricci-flat metric was first constructed in [5], and is contained within an extensive class of generalisations obtained by Stenzel in [11]. It achieves a “smoothing-out” of the apex of the cone metric, in which the manifold locally approaches $\mathbb{R}^3 \times S^3$ at the origin. The supergravity solution is expected to be dual to an $\mathcal{N} = 1$ supersymmetric $SU(N) \times SU(N + m)$ gauge theory [4, 10]. This theory will not have conformal invariance, since the large- r asymptotic structure of the fractional D3-brane solution is modified from the usual $H = 1 + Q/r^4$ form to $H = 1 + (Q + m^2 \log r)/r^4$. This modification to the leading-order fall-off behaviour is a consequence of the non-normalisability of the harmonic 3-form in the deformed conifold, which is used for supplying the R-R 3-form flux. As a consequence, the dual field theory is no longer conformal and the asymptotic behaviour of the gravitational solution for H correctly reproduces the asymptotic gauge coupling renormalisation of the two SYM factors [8, 9]. Related topics, and the implications of this type of solution, as applied to dual gauge theories in diverse dimensions, have been studied extensively in various papers [12, 13, 14, 15, 16, 17, 18, 19, 20, 21].

Solutions of a somewhat similar kind have been considered previously in different contexts. An M2-brane in which the transverse 8-manifold is still flat, but in which the 4-form field has an additional term proportional to a singular self-dual harmonic 4-form, was obtained in [22]. Solutions in which the 8-manifold is replaced by a complete Ricci-flat metric were discussed in [23]. Large classes of explicit solutions for a variety of cases, including M2-branes, D-branes, heterotic 5-branes and self-dual strings were constructed in [15, 18], with attention being focussed on obtaining deformations of the brane solutions that become regular everywhere. Some warped supergravity reductions related to the M2-brane examples were described in [24, 25, 26].

All of the solutions obtained in [15, 18] exploit the “transgression” terms of the form $dF_n = F_p \wedge F_q$ that occur in the Bianchi identities (or field equations) of certain field strengths in the associated supergravity theories. Specifically, the bilinear terms involving F_p and F_q are taken to be proportional to harmonic forms on the complete Ricci-flat space transverse to the original undeformed brane solution. Two types of solution can arise: (i) those where the flux integral of F_p or F_q is non-vanishing, and (ii) those where it vanishes. The first type gives rise to supergravity duals of fractional branes, with the non-vanishing flux corresponding to a wrapping of the additional brane around the cycle in the transverse manifold that is associated with the Hodge dual of the harmonic form. The second type of solution, with vanishing flux integrals, is not associated with fractional branes.¹ In the dual gauge-theory picture it was conjectured [18] that deformed solutions of this second type were related to the Higgs branch of the dual superconformal theory. In [20] a more extensive dual field theory interpretation of this second type of deformations was given and evidence was presented that they correspond to perturbations by relevant operators associated with the pseudo-scalar fields of a dual $N = 1$ superconformal theory. In addition, in [20], various kinds of fractional branes were discussed in the singular limit where the transverse manifold is a Ricci-flat cone. These are analogues of the fractional D3-brane conifold solution of [4]. They capture the correct large-distance asymptotic behaviour, but they become singular at short distance.

In this paper we shall focus on resolutions of various fractional branes, thus insisting that the resolved solution has a non-zero flux integral associated with the additional fields F_p

¹We are adopting the notation in this paper that any deformed solution in which there is a non-vanishing flux integral for an additional field, implying a wrapping of other branes around the associated homology cycle, will be called a “fractional brane,” by extension of the terminology for the fractional D3-brane, where D5-branes wrap around homology 2-cycles. By contrast, solutions without additional flux integrals will just be described as “deformed,” but not fractional.

and F_q . To obtain a regular resolution, one first needs to find a complete Ricci-flat manifold that can provide a smoothing-out of the apex of the cone. Then, it is necessary to find a suitable harmonic form in the complete manifold, which can give rise to deformed solutions with regular short-distance and large-distance behaviour. Specifically, the harmonic form should be square-integrable at short distance, and give a finite and non-vanishing flux at infinity.

The first cases we shall consider are resolved D2-branes, where the transverse space is 7-dimensional. One example of such a solution was obtained in [15], making use of a complete 7-dimensional manifold of G_2 holonomy that was obtained in [27, 28]. This manifold is of cohomogeneity one, with level surfaces that are topologically $S^3 \times S^3$. The manifold is the spin bundle of S^3 . Near the origin, it locally approaches $\mathbb{R}^4 \times S^3$. The deformed D2-brane solution has a non-vanishing flux for the NS-NS 5-brane, which wraps around the S^3 . Thus the solution can be viewed as a fractional NS-NS 2-brane, together with the usual D2-brane supported by the 4-form. The fractional flux is supported by an harmonic 3-form that is not L^2 normalisable, and the solution, while regular at small distance, corresponds at large distance to one with a linearly growing overall “charge.” The result indicates that asymptotically the renormalisation of $g_1^{-2} - g_2^{-2}$ may grow linearly with the energy scale, where g_1 and g_2 are the gauge couplings of the two SYM factors of the dual field theory. We shall discuss this solution in more detail in the present paper, in particular showing that it is supersymmetric, with 1/16 of the original supersymmetry preserved. Thus it corresponds to a dual three-dimensional $\mathcal{N} = 1$ field theory.

We shall also consider the second type of complete Ricci-flat 7-metrics of G_2 holonomy that were obtained in [27, 28]. They correspond to \mathbb{R}^3 bundles over four-dimensional quaternionic-Kähler Einstein base manifolds M . These spaces are again of cohomogeneity one, with level surfaces that are S^2 bundles over M (also known as “twistor spaces” over M). The two examples of four-dimensional quaternionic-Kähler space M that we shall consider are S^4 and \mathbb{CP}^2 . Thus the two manifolds have level surfaces that are \mathbb{CP}^3 (S^2 bundle over S^4) or the flag manifold $SU(3)/(U(1) \times U(1))$ (S^2 bundle over \mathbb{CP}^2), respectively. These two manifolds are the bundles of self-dual 2-forms over S^4 or \mathbb{CP}^2 respectively. They approach $\mathbb{R}^3 \times S^4$ or $\mathbb{R}^3 \times \mathbb{CP}^2$ locally near the origin.²

We shall use these manifolds in order to obtain new fractional D2-brane solutions. In these cases, the deformed D2-brane has a non-vanishing flux for a D4-brane, which wraps

²In what follows, the calculations for the two cases, with the principal orbits being S^2 bundles either over S^4 or over \mathbb{CP}^2 , proceed essentially identically. We shall in general therefore just refer to the S^2 bundle over S^4 example, with the understanding that all results apply, *mutatis mutandis*, to the other case too.

around the S^2 cycle. The solution can therefore be viewed as a fractional D2-brane, together with the usual D2-brane. In order to construct the solution we first need to obtain an harmonic 3-form on the 7-manifold. Although the construction is somewhat involved, we are able to obtain an explicit normalisable harmonic 3-form in this case. As a consequence the solution is not only regular at small distance, but also at large distance. At large distance, in the “decoupling limit,” it therefore approaches the metric of a regular D2-brane solution with Euclidean transverse space, and so the dual field theory in the ultraviolet regime approaches that of the regular D2-brane but with a different overall charge. We also show that this deformed solution is supersymmetric, preserving 1/16 of the original supersymmetry thus describing a dual three-dimensional $\mathcal{N} = 1$ field theory. The above two types of fractional D2-brane examples are discussed in section 2.

In section 3 we consider possible resolutions of a larger class of fractional branes, for which the singular cone configurations were discussed in [20]. Many of the explicit examples in [20] involved cones over compact manifolds \mathcal{M} that are themselves $U(1)$ bundles over products of complex projective spaces. (Examples include the 7-manifold $Q(1, 1, 1)$, which is a $U(1)$ bundle over $S^2 \times S^2 \times S^2$, and $M(2, 3)$, which is a $U(1)$ bundle over $S^2 \times \mathbb{CP}^2$.) In fact in [18], classes of complete Ricci-flat metrics were constructed that can provide resolutions of many such cone metrics. We make use of these here in order to discuss the possibility of obtaining regular fractional D-strings.

In section 4, we present a more extensive analysis of the resolutions of the cones for $U(1)$ bundles over products of Einstein-Kähler spaces, obtaining additional explicit complete Ricci-flat metrics, which go beyond those constructed in [18].

The paper ends with conclusions in section 5.

2 Fractional D2-branes

By making use of the transgression term in the equation of motion $d(e^{\frac{1}{2}\phi} \hat{*} F_4) = F_{(4)} \wedge F_{(3)}$, one can construct the a deformed D2-brane solution of type IIA supergravity, which is given by [15]

$$\begin{aligned} ds_{10}^2 &= H^{-5/8} dx^\mu dx^\nu \eta_{\mu\nu} + H^{3/8} ds_7^2, \\ F_{(4)} &= d^3x \wedge dH^{-1} + m G_{(4)}, \quad F_{(3)} = m G_{(3)}, \quad \phi = \frac{1}{4} \log H, \end{aligned} \quad (2.1)$$

where $G_{(3)}$ is an harmonic 3-form in the Ricci-flat 7-metric ds_7^2 , and $G_{(4)} = *G_{(3)}$, with $*$ the Hodge dual with respect to ds_7^2 . The function H satisfies

$$\square H = -\frac{1}{6} m^2 G_{(3)}^2, \quad (2.2)$$

where \square denotes the scalar Laplacian with respect to the transverse 7-metric ds_7^2 . Thus the deformed D2-brane solution is completely determined by the choice of Ricci-flat 7-manifold, and the harmonic 3-form.

The easiest way to determine whether the deformed solution is supersymmetric is to lift it first to $D = 11$, and then to examine the supersymmetry of the resulting solution of eleven-dimensional supergravity. Using the standard Kaluza-Klein rules, the metric in (2.1) becomes

$$d\hat{s}_{11}^2 = H^{-2/3} dx^\mu dx^\nu \eta_{\mu\nu} + H^{1/3} ds_8^2, \quad (2.3)$$

where

$$ds_8^2 = ds_7^2 + dz^2 \quad (2.4)$$

and z is the eleventh coordinate. The 4-form in $D = 11$ is given by

$$\hat{F}_4 = d^3x \wedge dH^{-1} + m \hat{G}_{(4)}, \quad (2.5)$$

where

$$\hat{G}_{(4)} = G_{(4)} + G_{(3)} \wedge dz, \quad (2.6)$$

where, as in (2.1), $G_{(4)} = *G_{(3)}$ and $*$ still means the seven-dimensional dual. Thus in $D = 11$ we have a “resolved M2-brane,” within the general class discussed in [15], where $\hat{G}_{(4)}$ is a 4-form that is harmonic and self-dual with respect to the Ricci-flat 8-dimensional metric (2.5). Of course this 8-metric is not asymptotically conical, since the eighth direction provides just an \mathbb{R} factor. Nonetheless, we can apply all the standard $D = 11$ resolved-brane formulae for testing the supersymmetry.

In [23, 25, 15], it is shown that the additional requirement for supersymmetry when the harmonic 4-form $\hat{G}_{(4)}$ is turned on is

$$\hat{G}_{abcd} \Gamma_{bcd} \eta = 0, \quad (2.7)$$

where η is a covariantly-constant spinor in the Ricci-flat 8-metric. Thus, in order to check the supersymmetry of the deformed D2-brane solution, we simply have to lift the harmonic 3-form $G_{(3)}$ in the Ricci-flat 7-metric using (2.6), to get a self-dual harmonic 4-form in the Ricci-flat 8-metric (2.4), and then check the integrability condition (2.7).

In the following subsections, we shall consider various choices for the Ricci-flat 7-manifold. We begin with examples using Ricci-flat cones, which are singular at the apex. Although the fractional D2-brane solutions will be singular, it is useful to study these first since they capture the large-distance behaviour that will arise in resolved versions of these manifolds. Then, we consider the two types of resolved examples using complete Ricci-flat G_2 manifolds; firstly, the S^2 bundles over S^4 or \mathbb{CP}^2 , and then the S^3 bundle over S^3 .

2.1 Fractional D2-branes over Ricci-flat cones

Here, we take the Ricci-flat 7-metric to be

$$ds_7^2 = dr^2 + r^2 d\Sigma_6^2, \quad (2.8)$$

where $d\Sigma_6^2$ is the metric on a compact Einstein 6-manifold M_6 satisfying $R_{ab} = 5 g_{ab}$. As discussed in [20], one can obtain a fractional D2-brane solution if M_6 has a non-trivial 4-cycle around whose dual 2-cycle a D4-brane can wrap. If $\omega_{(4)}$ denotes the associated harmonic 4-form in M_6 we can set $G_{(4)} = \omega_{(4)}$ in (2.1) in order to obtain a solution. One finds that the function H is given, for a suitable normalisation for $\omega_{(4)}$, by [20]

$$H = c_0 + \frac{Q}{r^5} - \frac{m^2}{4r^6}. \quad (2.9)$$

The fractional D2-brane carries an electric charge Q and a magnetic charge m for $F_{(4)}$, while the 3-form, given by $F_{(3)} = m r^{-4} dr \wedge *_6 \omega_{(4)}$, gives no flux integral.

The solution is analogous to the original fractional D3-brane on the 6-dimensional conifold, constructed in [4], and it also suffers from a naked singularity at some positive value of r where H vanishes.

A possible choice for M_6 would be \mathbb{CP}^3 , in which case $\omega_{(4)}$ is just given by $J \wedge J$, where J is the Kähler form of \mathbb{CP}^3 . In fact the cone over a manifold of \mathbb{CP}^3 topology admits a smooth resolution, and in section 2.2 below, we shall use this to obtain a completely regular fractional D2-brane with wrapped D4-brane.

A completely different kind of “fractional” 2-brane can be obtained if M_6 has a non-trivial 3-cycle, around whose dual 3-cycle a 5-brane can wrap. In this case, we let $G_{(3)} = \omega_{(3)}$ in (2.1), where $\omega_{(3)}$ is the harmonic form on M_6 associated with the 3-cycle. The function H is now given by

$$H = c_0 + \frac{m^2}{4r^4} + \frac{Q}{r^5}. \quad (2.10)$$

Since there is now a term with the slower fall-off $1/r^4$, the solution no longer has a well-defined ADM mass. If one nevertheless continues to define a 4-form “electric charge” to be proportional to $r^6 H'$, then this gives the r -dependent result

$$\text{“charge”} = Q + \frac{m^2}{5} r, \quad (2.11)$$

where m is the magnetic 5-brane charge carried by $F_{(3)}$. The “electric charge” (2.11) depends linearly on the distance, and this feature, and the associated ill-definement of the ADM mass, is analogous to the logarithmic dependence of the fractional D3-brane charge in [4].

The solution can be viewed as the usual D2-brane, together with a fractional NS-NS 2-brane that arises as an NS-NS 5-brane wrapped around the 3-cycle.

This solution is singular at $r = 0$, which coincides with the horizon. A possible choice for M_6 is $S^3 \times S^3$, with $\omega_{(3)}$ taken to be one of the volume forms of an S^3 factor. If the two S^3 factors formed a direct product then the solution would be unsatisfactory for a variety of reasons, including the absence of supersymmetry and also that it would not be resolvable. However, one can instead take an S^3 bundle over S^3 , and although the bundle is trivial, implying that it is still topologically $S^3 \times S^3$, the metric will no longer be a direct sum. This allows a resolution to a complete Ricci-flat 7-manifold, with G_2 holonomy, and in fact it was one of the examples constructed in [27, 28]. Topologically, the 7-manifold is the spin bundle of S^3 , which is $\mathbb{R}^4 \times S^3$. The corresponding resolved D2-brane was constructed in [15]. We shall discuss this further in section 2.3.

An interesting feature of both the resolutions that we shall consider below is that they cause the original D2-brane charge parameter Q to become related to the parameter m characterising the charge of the wrapped branes. Thus although Q and m are independent in the cone metric, they become correlated in the resolutions. In fact, this feature arises too in the case of the deformed fractional D3-brane in [10].

2.2 Regular fractional D2-brane

In this section we consider a resolution of the fractional D2-brane on the cone over \mathbb{CP}^3 . To do this, we need to describe the relevant complete Ricci-flat metric, and also to construct a suitable harmonic 3-form on the 7-manifold.

The complete Ricci-flat 7-metric on the bundle of self-dual 2-forms over S^4 was constructed in [27, 28]. In the notation of [28], it is given by

$$d\hat{s}_7^2 = h^2 dr^2 + a^2 (D\mu^i)^2 + b^2 d\Omega_4^2, \quad (2.12)$$

where μ^i are coordinates on \mathbb{R}^3 subject to $\mu^i \mu^i = 1$, $d\Omega_4^2$ is the metric on the unit 4-sphere, with $SU(2)$ Yang-Mills instanton potentials A^i , and

$$D\mu^i \equiv d\mu^i + \epsilon_{ijk} A^j \mu^k. \quad (2.13)$$

The field strengths $J^i \equiv dA^i + \frac{1}{2}\epsilon_{ijk} A^j \wedge A^k$ satisfy the algebra of the unit quaternions, $J_{\alpha\gamma}^i J_{\gamma\beta}^j = -\delta_{ij} \delta_{\alpha\beta} + \epsilon_{ijk} J_{\alpha\beta}^k$. A convenient orthonormal basis is

$$\hat{e}^0 = h dr, \quad \hat{e}^i = a D\mu^i, \quad \hat{e}^\alpha = b e^\alpha. \quad (2.14)$$

(Note that although i runs over 3 values, there are really only two independent vielbeins on the 2-sphere, because of the constraint $\mu^i \mu^i = 1$.)

The metric is Ricci-flat, with G_2 holonomy, when the functions h , a and b are given by [28]

$$h^2 = \left(1 - \frac{1}{r^4}\right)^{-1}, \quad a^2 = \frac{1}{4}r^2 \left(1 - \frac{1}{r^4}\right), \quad b^2 = \frac{1}{2}r^2. \quad (2.15)$$

The radial coordinate runs from $r = 1$, where the metric locally approaches $\mathbb{R}^3 \times S^4$, to the asymptotically-flat region at $r = \infty$. The principal orbits at fixed r are \mathbb{CP}^3 , described as an S^2 bundle over S^4 . (This is the *ur* twistor space.) Thus this metric provides a resolution of the Ricci-flat cone over \mathbb{CP}^3 . It should be noted, however, that the metric at large distance is asymptotic to the cone over the “squashed” Einstein metric on \mathbb{CP}^3 and not the Fubini-Study Einstein metric.³

Before considering fractional D2-branes, we can first look for the usual type of D2-brane solution but in the background of this Ricci-flat transverse metric. We find that the radially-symmetric solution to $\square H = 0$ is given by

$$H = c_0 + \frac{1}{r(1 - r^{-4})^{1/2}} - F(\arcsin(\frac{1}{r})| - 1), \quad (2.16)$$

where

$$F(\phi|m) \equiv \int_0^\phi (1 - m \sin^2 \theta)^{-1/2} d\theta \quad (2.17)$$

is the incomplete elliptic integral of the first kind. This approaches a constant plus $O(r^{-5})$ at large r , and diverges as $1/\sqrt{r-1}$ near $r = 1$. Thus the D2-brane metric has a singularity at $r = 1$, which coincides with the horizon.

Now let us look for a fractional D2-brane, which requires finding a suitable harmonic 3-form, which is L^2 -integrable at short distance and whose dual 4-form has a non-vanishing flux integral at infinity. In fact, as we shall see below, we are able to obtain a fully L^2 -normalisable harmonic 3-form in this example.⁴ As in [28], we can make the following ansatz for the harmonic 3-form:

$$G_{(3)} = f_1 dr \wedge X_{(2)} + f_2 dr \wedge J_{(2)} + f_3 X_{(3)}, \quad (2.18)$$

where

$$X_{(2)} \equiv \frac{1}{2} \epsilon_{ijk} \mu^i D\mu^j \wedge D\mu^k, \quad J_{(2)} \equiv \mu^i J^i, \quad X_{(3)} \equiv D\mu^i \wedge J^i. \quad (2.19)$$

³The usual and the squashed Einstein metrics are the members of the family of \mathbb{CP}^3 metrics $ds_6^2 = \lambda^2 (D\mu^i)^2 + d\Omega_4^2$ with $\lambda^2 = 1$ and $\lambda^2 = 1/2$ respectively [28]. The squashed Einstein metric is Hermitean but not Kähler. In fact, it is nearly Kähler (see, for example, [29].)

⁴This accords with the fact that above the middle dimension (and hence, by Hodge duality, below the middle dimension) the L^2 cohomology should be topological, and thus isomorphic to ordinary compactly-supported cohomology. We are grateful to Nigel Hitchin for discussion on this point.

One can easily see that

$$dX_{(2)} = X_{(3)}, \quad dJ_{(2)} = X_{(3)}, \quad dX_{(3)} = 0. \quad (2.20)$$

Imposing the harmonicity conditions $dG_{(3)} = 0$ and $d*G_{(3)} = 0$, we obtain the equations (correcting some typographical errors in [28])

$$f'_3 = f_1 + f_2, \quad \left(\frac{f_1 b^4}{h a^2}\right)' = 4h f_3, \quad \left(\frac{f_2 a^2}{h}\right)' = 2h f_3. \quad (2.21)$$

It is useful at this stage to note that from the covariantly-constant spinor η that exists in this manifold, we can build a covariantly-constant 3-form which must, therefore, satisfy the equations (2.21). From results in [28], one can show that this has $f_1 = h a^2$, $f_2 = h b^2$ and $f_3 = a b^2$, and that this does indeed satisfy (2.21). This motivates the following field redefinitions, namely

$$f_1 = h a^2 u_1, \quad f_2 = h b^2 u_2, \quad f_3 = a b^2 u_3. \quad (2.22)$$

Note that this means that the u_i functions are the coefficients of wedge-products of the hatted vielbeins for the 3-form

$$G_{(3)} = \frac{1}{2} u_1 \mu^i \epsilon_{ijk} \hat{e}^0 \wedge \hat{e}^j \wedge \hat{e}^k + \frac{1}{2} u_2 \mu^i J_{\alpha\beta}^i \hat{e}^0 \wedge \hat{e}^\alpha \wedge \hat{e}^\beta + \frac{1}{2} J_{\alpha\beta}^i u_3 \hat{e}^i \wedge \hat{e}^\alpha \wedge \hat{e}^\beta. \quad (2.23)$$

The coordinate transformation $x = r^4$ puts the equations into the form

$$\begin{aligned} 4x(x-1) \frac{du_3}{dx} + (3x-1)u_3 - 2xu_2 - (x-1)u_1 &= 0, \\ \frac{d(xu_1)}{dx} = u_3, \quad \frac{d((x-1)u_2)}{dx} &= u_3. \end{aligned} \quad (2.24)$$

From the last two we can obtain the first integral

$$(x-1)u_2 - xu_1 = c_0, \quad (2.25)$$

where c_0 is a constant.

It is now straightforward to obtain an equation purely for u_1 . It is advantageous to make a further coordinate transformation, $x = 1/y$, and to define $u_1 = v_1 - c_0$, after which we find

$$4y(1-y)^2 \ddot{v}_1 - (1-y)(3-y) \dot{v}_1 - 2v_1 = 0, \quad (2.26)$$

where a dot means d/dy . The solution can be expressed in terms of elliptic integrals. Retracing the steps, we finally arrive at the following expressions for u_1 , u_2 and u_3 in terms

of the original r coordinate:

$$\begin{aligned} u_1 &= \frac{1}{r^4} + \frac{P(r)}{r^5 (r^4 - 1)^{1/2}}, \\ u_2 &= -\frac{1}{2(r^4 - 1)} + \frac{P(r)}{r (r^4 - 1)^{3/2}}, \\ u_3 &= \frac{1}{4r^4 (r^4 - 1)} - \frac{(3r^4 - 1) P(r)}{4r^5 (r^4 - 1)^{3/2}}, \end{aligned} \quad (2.27)$$

where

$$P(r) = F(\tfrac{1}{2}\pi | -1) - F(\arcsin(r^{-1}) | -1) = \int_{\arcsin(r^{-1})}^{\frac{1}{2}\pi} \frac{d\theta}{\sqrt{1 + \sin^2 \theta}}. \quad (2.28)$$

Note that the functions u_i satisfy $u_1 + 2u_2 + 4u_3 = 0$.

We have made appropriate choices for the integration constants, in order to ensure that the functions u_i are non-singular at $r = 1$, and that they fall off at large r . Near $r = 1$, the functions have the following behaviour:

$$\begin{aligned} u_1 &= \frac{3}{2} - 7(r - 1) + \frac{203}{10}(r - 1)^2 + \dots, \\ u_2 &= -\frac{1}{4} + \frac{7}{10}(r - 1) - \frac{23}{20}(r - 1)^2 + \dots, \\ u_3 &= -\frac{1}{4} + \frac{7}{5}(r - 1) - \frac{9}{2}(r - 1)^2 + \dots. \end{aligned} \quad (2.29)$$

At large r , the asymptotic behaviour is

$$\begin{aligned} u_1 &= \frac{1}{r^4} + \frac{\gamma_0}{r^7} - \frac{1}{r^8} + \dots, \\ u_2 &= -\frac{1}{2r^4} + \frac{\gamma_0}{r^7} - \frac{3}{2r^8} + \dots, \\ u_3 &= -\frac{3\gamma_0}{4r^7} + \frac{1}{r^8} + \dots, \end{aligned} \quad (2.30)$$

where $\gamma_0 = F(\frac{1}{2}\pi | -1) = \sqrt{\pi} \Gamma(\frac{5}{4}) / \Gamma(\frac{3}{4}) = 1.3110\dots$

The magnitude of $G_{(3)}$ is given by

$$|G_{(3)}|^2 = 6(u_1^2 + 2u_2^2 + 6u_3^2). \quad (2.31)$$

It is evident from (2.23), and the asymptotic forms (2.29) and (2.30), that $|G_{(3)}|^2$ tends to a constant at small r , and $|G_{(3)}|^2 \sim 1/r^8$ at large r . Since the metric has $\sqrt{g} \sim r^6$ at large r , it follows that the harmonic 3-form is L^2 -normalisable.

We can also examine the flux associated with this harmonic form. Taking its Hodge dual with respect to the 7-metric, we find

$$G_{(4)} \equiv *G_{(3)} = \frac{1}{4}u_1 r^4 \Omega_{(4)} + \dots, \quad (2.32)$$

where $\Omega_{(4)}$ is the volume form of the unit 4-sphere. We then see from (2.30) that the integral of $\mathcal{G}_{(4)}$ over the 4-sphere at infinity gives a finite and non-vanishing charge

$$P_4 \equiv \frac{1}{V(S^4)} \int G_{(4)} = \frac{1}{4}. \quad (2.33)$$

This implies that our D2-brane solution carries additional wrapped D4-brane charge.

Having obtained the harmonic 3-form $G_{(3)}$, we can substitute into (2.1) and obtain the solution for the resolved fractional D2-brane. It is difficult to give a fully explicit expression for H , since the harmonic 3-form itself has a rather complicated expression. It is easy to find the asymptotic forms for H at large distance, and at the origin (*i.e.* at $r = 1$). They are given by

$$\begin{aligned} \text{Large distance :} \quad H &= c_0 + \frac{Q}{r^5} - \frac{m^2}{4r^6} + \dots, \\ \text{Short distance :} \quad H &= c_1 - \frac{11m^2(r-1)}{24} + \dots, \end{aligned} \quad (2.34)$$

where

$$Q = \frac{m^2}{30} \int_1^\infty dr r^4 \sqrt{r^4 - 1} |G_{(3)}|^2. \quad (2.35)$$

The behaviour at large r is indeed the same as in the case of the cone metric, given by (2.9). However, now we see that the short-distance behaviour is quite different from (2.9), and in fact the metric is regular there. Note that the D2-brane charge Q , which was a free parameter in the cone metric solution discussed in section 2.1, is now quadratically dependent on the charge m carried by the wrapped D4-brane.

To test the supersymmetry of this resolved D2-brane solution, we apply the procedure described at the beginning of section 2, of first lifting it to $D = 11$. The vielbein components of the harmonic 3-form in the Ricci-flat 7-metric can be read off from (2.23). For the purposes of the present purely algebraic calculation, a convenient way to handle the fact that there are just two, rather than three, independent vielbeins \hat{e}^i , owing to the constraint $\mu_i \mu_i = 1$, is to make a specific choice, such as $\mu_i = (0, 0, 1)$, so that we shall have only \hat{e}^1 and \hat{e}^2 non-zero. If we let the indices α in the S^4 base range of $\alpha = (3, 4, 5, 6)$, then by making a basis choice for the quaternionic Kähler forms, such as

$$J_{34}^1 = J_{56}^1 = J_{35}^2 = J_{64}^2 = J_{36}^3 = J_{45}^3 = -1, \quad (2.36)$$

then we read off from (2.23) that the non-vanishing vielbein components of $G_{(3)}$ are given by

$$G_{012} = u_1, \quad G_{045} = G_{036} = -u_2, \quad G_{156} = G_{134} = G_{264} = G_{235} = -u_3. \quad (2.37)$$

Using (2.6), we then easily read off the 8-dimensional vielbein components of the harmonic self-dual 4-form $\hat{G}_{(4)}$.

From results in [28], the covariantly-constant spinor η in the Ricci-flat 7-metric satisfies integrability conditions that are completely specified by $(4\Gamma_{01} - J_{\alpha\beta}^2 \Gamma_{\alpha\beta}) \eta = 0$ and $(4\Gamma_{02} + J_{\alpha\beta}^1 \Gamma_{\alpha\beta}) \eta = 0$ (after making our convenient local basis choice $\mu_i = (0, 0, 1)$). This translates into

$$(2\Gamma_{01} + \Gamma_{64} + \Gamma_{35}) \eta = (2\Gamma_{02} - \Gamma_{56} - \Gamma_{34}) \eta = 0. \quad (2.38)$$

Substituting the components for $\hat{G}_{(4)}$, obtained as described above, into (2.7), we find after using (2.38) that the condition for supersymmetry is satisfied provided that

$$u_1 + 2u_2 + 4u_3 = 0. \quad (2.39)$$

As noted earlier, the functions u_i , obtained in (2.27), do indeed satisfy this condition, and so we conclude that this resolved fractional D2-brane solution is supersymmetric. To be precise, we mean by this that turning on the contribution of the harmonic 3-form leads to no further loss of supersymmetry, over and above the reduction that already occurred when the flat transverse 7-space of the usual D2-brane was replaced by the Ricci-flat manifold of G_2 holonomy. Thus we have obtained a completely regular supersymmetric solution describing the usual D2-brane together with a fractional D2-brane coming from the wrapping of D4-branes around a 2-cycle. As we remarked in footnote 1, we can also get a completely analogous fractional D2-brane in which the Ricci-flat 7-manifold is replaced by the related example in [27, 28], where the principal orbits are S^2 bundles over \mathbb{CP}^2 instead of S^4 . We refer to [28] for further details of this example, and for the analogous conditions on the covariantly-constant spinor, which again lead to the conclusion that the associated fractional D2-brane will preserve supersymmetry as above.

To summarise, owing to the existence of a normalisable harmonic 3-form on these Ricci-flat 7-manifolds with the topology of \mathbb{R}^3 bundles over S^4 or \mathbb{CP}^2 , the corresponding fractional D2-brane solutions are regular both at small distance as well as at large distance. Thus, in the decoupling limit they have the same asymptotic large-distance behaviour as regular D2-branes with Euclidean transverse spaces. As a consequence, the dual asymptotic field theory is that of a regular D2-brane (whose original charge $Q \sim N$ determines the SYM factor $SU(N)$), but now with a different overall charge $Q \propto m^2 \sim M$, which is related to the contribution of the additional fluxes of the wrapped D4-branes, and thus indicating a single SYM factor $SU(M)$. These deformed solutions preserve 1/16 of the original supersymmetry thus describing a dual three-dimensional $\mathcal{N} = 1$ field theory.

2.3 Regular fractional NS-NS 2-brane

The resolved D2-brane using the 7-manifold of G_2 holonomy corresponding to the spin bundle of S^3 was constructed in [15], but its supersymmetry was not analysed there. Here, we summarise the results, and then we shall show that it is supersymmetric. The Ricci-flat metric on the spin bundle of S^3 is given by [27, 28]

$$ds_7^2 = \alpha^2 dr^2 + \beta^2 (\sigma_i - \frac{1}{2}\Sigma_i)^2 + \gamma^2 \Sigma_i^2, \quad (2.40)$$

where the functions α , β and γ are given by

$$\alpha^2 = \left(1 - \frac{1}{r^3}\right)^{-1}, \quad \beta^2 = \frac{1}{9}r^2 \left(1 - \frac{1}{r^3}\right), \quad \gamma^2 = \frac{1}{12}r^2. \quad (2.41)$$

Here Σ_i and σ_i are two sets of left-invariant 1-forms on two independent $SU(2)$ group manifolds. The level surfaces $r = \text{constant}$ are therefore S^3 bundles over S^3 . This bundle is trivial, and so in fact the level surfaces are topologically $S^3 \times S^3$. The radial coordinate runs from $r = a$ to $r = \infty$. We define an orthonormal frame by

$$e^0 = \alpha dr, \quad e^i = \gamma \Sigma_i, \quad \tilde{e}^i = \beta \nu_i \quad (2.42)$$

where $\nu_i \equiv \sigma_i - \frac{1}{2}\Sigma_i$.

The metric represents a smoothing out of a cone over $S^3 \times S^3$. However, it should be noted that at large distance the principal orbits approach the “squashed” Einstein metric on $S^3 \times S^3$ rather than the usual one. These are both members of the homogeneous family of $S^3 \times S^3$ metrics $ds_6^2 = \lambda^2 \nu_i^2 + \Sigma_i^2$, with $\lambda^2 = 4/3$ and $\lambda^2 = 4$ respectively (see, for example, [28]). Again, the squashed Einstein metric is nearly Kähler, although not Kähler (see, for example, [29]).

Before presenting the fractional D2-brane solution that was obtained in [15], we may first look at the usual D2-brane solution, but in this Ricci-flat seven-dimensional transverse-space background. The radially-symmetric solution to the equation $\square H = 0$ is

$$H = c_0 - \frac{3r}{r^3 - 1} + \log \left[\frac{r^2 + r + 1}{(r - 1)^2} \right] + 2\sqrt{3} \arctan \left[\frac{2r + 1}{\sqrt{3}} \right]. \quad (2.43)$$

At large r this approaches a constant plus $O(r^{-5})$. Near to $r = 1$, there are divergences of order $1/(r - 1)$ and $\log(r - 1)$. As in the previous example, the metric in the absence of fractional branes has a singularity at $r = 1$, which coincides with the horizon.

It was shown in [15] that the metric (2.40) admits an harmonic 3-form given by

$$G_{(3)} = v_1 e^0 \wedge \tilde{e}^i \wedge e^i + v_2 \epsilon_{ijk} \tilde{e}^i \wedge \tilde{e}^j \wedge e^k + \frac{1}{6} v_3 \epsilon_{ijk} e^i \wedge e^j \wedge e^k, \quad (2.44)$$

where the functions v_1 , v_2 and v_3 are given by

$$v_1 = -\frac{(3r^2 + 2r + 1)}{r^4(r^2 + r + 1)^2}, \quad v_2 = \frac{(r^2 + 2r + 3)}{r(r^2 + r + 1)^2}, \quad v_3 = \frac{3(r + 1)(r^2 + 1)}{r^4(r^2 + r + 1)}. \quad (2.45)$$

It should be noted that these satisfy the relation

$$3v_1 - 3v_2 + v_3 = 0. \quad (2.46)$$

This harmonic 3-form is square-integrable at short distance, but it gives a linearly divergent integral at large distance. As shown in [15], the short-distance square-integrability is enough to give a perfectly regular deformed D2-brane solution, even though $G_{(3)}$ is not L^2 -normalisable. We may also note that it gives a finite and non-vanishing flux, when integrated over the 3-sphere associated with the metric Σ_i^2 at infinity, since we have

$$G_{(3)} = \frac{1}{3\sqrt{3}} r^3 v_3 \Omega_{(3)} + \dots, \quad (2.47)$$

leading to a charge

$$P_3 \equiv \frac{1}{V(S^3)} \int G_{(3)} = \frac{1}{\sqrt{3}}. \quad (2.48)$$

This implies that the D2-brane solution carries additional wrapped NS-NS 5-brane charge.

The solution for the function H in the corresponding deformed D2-brane solution (2.1) was shown to be given by [15]

$$\begin{aligned} H = & c_0 + \frac{m^2(r+1)(16r^7 + 24r^6 + 48r^5 + 47r^4 + 54r^3 + 36r^2 + 18r + 9)}{108r^3(r^2 + r + 1)^3} \\ & + \frac{8m^2}{27\sqrt{3}} \arctan\left[\frac{2r+1}{\sqrt{3}}\right], \end{aligned} \quad (2.49)$$

(after making a rescaling $G_{(3)} \rightarrow G_{(3)}/108$ for convenience). The function H is perfectly non-singular for r running from the origin at 1 to infinity. At small distance and large distance it has the forms

$$\begin{aligned} r \rightarrow 1 : \quad & H = c_0 + \frac{m^2}{27} \left[\frac{2}{3} \left(7 - \frac{2\pi}{\sqrt{3}} \right) - 7(r-1) + 14(r-1)^2 + \dots \right], \\ r \rightarrow \infty : \quad & H = c_0 + \frac{m^2}{4r^4} - \frac{4m^2}{15r^5} + \frac{9m^2}{28r^7} + \dots. \end{aligned} \quad (2.50)$$

The electric charge for the 4-form supporting the usual D2-brane is given by

$$\text{“charge”} = -\frac{4m^2}{15} + \frac{m^2}{5} r. \quad (2.51)$$

As expected, this exhibits the same linear dependence on r as we saw for the cone metric in section 2.1. However, the resolution determines that the parameter Q in (2.11), which was arbitrary for the cone metric, is now given in terms of the parameter m by $Q = -4m^2/15$.

The linear growth of the overall “charge” indicates an asymptotic dual field theory with two SYM factors (which are now modified due to the fractional NS-NS 2-brane contribution), and the renormalisation of the difference of the coupling couplings may grow linearly with the energy scale.

Note also, that the small distance behaviour of the above solution is qualitatively the same as that of the fractional D2-brane discussed in the previous subsection, thus indicating the same universal infrared behaviour of dual field theories for both types of solutions.

We shall now show that the solution preserves 1/16 of the original supersymmetry, thus describing a regular supergravity dual of a three-dimensional $\mathcal{N} = 1$ field theory. It was shown in [28] that the covariantly-constant spinor in this 7-metric of G_2 holonomy satisfies integrability conditions that are completely specified by

$$(\Gamma_{04} - \Gamma_{23})\eta = (\Gamma_{05} - \Gamma_{31})\eta = (\Gamma_{06} - \Gamma_{12})\eta = 0. \quad (2.52)$$

Following the same procedures as in the previous subsection, we now lift the harmonic 3-form (2.44) to $D = 8$ using (2.6), and then substitute into the supersymmetry criterion (2.7). After making use of (2.52), we find that supersymmetry is indeed preserved by virtue of the linear relation (2.46) amongst the functions v_1 , v_2 and v_3 appearing in (2.44).

Thus we have established that the completely regular resolved D2-brane solution obtained in [15] is supersymmetric, and that it describes the usual D2-brane together with fractional NS-NS 2-branes coming from the wrapping of 5-branes around a 3-cycle.

2.4 Spectrum analysis

Having obtained the non-singular supergravity solutions for fractional D2-branes that are dual to the $\mathcal{N} = 1$ gauge theories in $D = 3$, it is of interest to study the spectrum of the minimally coupled scalar wave equation in these supergravity backgrounds. In particular, by examining whether the spectrum is discrete, one can see whether the gauge theory exhibits confinement in the infrared regime.

Clearly, one cannot expect to be able to solve the wave equations explicitly for these solutions, owing to their complexity. However, the characteristics of the spectrum can be determined from the structure of the Schrödinger potential for the minimally-coupled scalar wave equation. A simple way to obtain the Schrödinger potential is to reduce the solutions on the principal orbits so that they become $D = 4$ domain walls. The resulting 4-dimensional metric can be cast into a conformal frame, of the form

$$ds_4^2 = e^{2A(z)} (dx^\mu dx_\nu \eta_{\mu\nu} + dz^2). \quad (2.53)$$

The Schrödinger potential is then given by

$$V(z) = A'' + (A')^2. \quad (2.54)$$

For both our fractional D2-brane solutions, the coordinate z runs from 0 to a finite value z^* . The behaviours of $A(z)$ and $V(z)$ near $z = 0$ and $z = z^*$ are given by

$$\begin{aligned} &\underline{\text{Fractional D2-brane :}} \\ &r \longrightarrow \infty, \quad z \longrightarrow 0 : \quad e^{2A} \sim \frac{1}{z^{7/3}}, \quad V \sim \frac{91}{36z^2} \\ &r \longrightarrow 1, \quad z \longrightarrow z^* : \quad e^{2A} \sim z^2, \quad V \sim 0, \\ &\underline{\text{Fractional NS-NS 2-brane :}} \\ &r \longrightarrow \infty, \quad z \longrightarrow 0 : \quad e^{2A} \sim \frac{1}{z^4}, \quad V \sim \frac{6}{z^2} \\ &r \longrightarrow 1, \quad z \longrightarrow z^* : \quad e^{2A} \sim z^3, \quad V \sim \frac{3}{4(z - z^*)^2}. \end{aligned} \quad (2.55)$$

(These correspond to the G_2 holonomy 7-metrics for the bundle of self-dual 2-forms over S^4 , and the spin bundle of S^3 , respectively.) Thus both Schrödinger potentials are of infinite-well type, and so the spectra for both solutions are discrete, indicating confinement for each of the corresponding dual $\mathcal{N} = 1$, $D = 3$ gauge theories. Again, the results of this subsection indicate that the spectra for both cases have qualitatively the same behaviour.

3 Resolution of other fractional branes

As we discussed in the introduction, fractional branes were constructed by making use of the Bianchi identity (or the equation of motion) $dF_n = F_p \wedge F_q$. The solution has a transverse space that is a complete, non-compact Ricci-flat manifold of dimension $n + 1 = p + q$. In order for F_n to carry point-like charge (as opposed to a distribution of charge), the Ricci-flat metric at large distance should approach the conical form

$$ds_{n+1}^2 = dr^2 + r^2 d\Sigma_n^2, \quad (3.1)$$

where $d\Sigma_n^2$ is an Einstein metric (with $R_{ab} = (n - 1)g_{ab}$) on a compact n -dimensional manifold M_n . Furthermore, if either F_p or F_q is to have a non-vanishing flux, then M_n must have either a non-trivial p -cycle or q -cycle. Without loss of generality, let us consider the case of a p -cycle, for which F_p will carry the non-trivial flux provided by an harmonic p -form. This can always be achieved by choosing $M_n = S^p \times S^{q-1}$. However, this configuration will not in general be supersymmetric, and in fact supersymmetric backgrounds of this kind are uncommon.

The large-distance behaviour of the metric function H in the fractional branes is, however, universal, given for $p \neq 1$ by

$$H \sim \frac{Q}{r^{n-1}} + \frac{m^2}{(q-p)(p-1)r^{2p-2}} = \frac{Q}{r^{n-1}} \left(1 + \frac{m^2/Q}{(q-p)(p-1)r^{p-q}} \right), \quad (3.2)$$

and for $p = 1$ by

$$H \sim \frac{Q}{r^{n-1}} - \frac{m^2 \log r}{(q-1)}. \quad (3.3)$$

Solutions with $p = 0$ and $p = 1$ would have naked singularities at large distance, at the radius where H vanished, and so it seems necessary to consider cases with $p \geq 2$. For these, if $p > q$ the solutions have a well-defined ADM mass and the flux for F_n is constant. In other words, there is no contribution from the $F_p \wedge F_q$ transgression term to the n -form flux. If $p \leq q$, the solutions do not have a well-defined ADM mass, the transgressions terms contribute to the n -form flux, and furthermore it is r -dependent. In particular, if $p = q$ the dependence is logarithmic, since now (3.2) becomes

$$H \sim \frac{Q + m^2 \log r}{r^{n-1}}. \quad (3.4)$$

For $p \geq 2$, all the fractional branes are well behaved at large distance. If one takes the cone solutions as they are, they all suffer from singularities at small distance. In particular, for $p \geq q$ the singularity is naked, at some positive value of r , whilst for $p < q$ the singularity coincides with the horizon at $r = 0$.

In order to avoid this singularity, it is necessary that the harmonic p -form be L^2 normalisable at short distance, in the $(n+1)$ -dimensional Ricci-flat manifold [15, 18]. This implies that the Ricci-flat metric should interpolate between the cone metric at large distance and the following metric at small distance:

$$ds_{n+1}^2 = d\rho^2 + \rho^2 d\Sigma_s^2 + d\Sigma_{q-s-1}^2 + d\Sigma_p^2, \quad (3.5)$$

where s can take some value in the range 1 to $q-1$. In other words, there should be a non-collapsing p -surface. Indeed the resolved fractional D3-brane in [10] and the D2-branes we discussed in the previous section satisfy these criteria.

In [14, 21], alternative resolutions of the $T^{1,1}$ conifold were considered, for which the short-distance behaviour of the Ricci-flat 6-metric is

$$ds_6^2 = d\rho^2 + \rho^2 d\nu^2 + (\rho^2 + \ell_1^2) d\Omega_1^2 + (\rho^2 + \ell_2^2) d\Omega_2^2, \quad (3.6)$$

where $d\Omega_1^2$ and $d\Omega_2^2$ are two 2-sphere metrics, and $d\nu = \Omega_1 + \Omega_2$, the sum of the volume forms of the two 2-sphere metrics. In this resolution there is no non-collapsing 3-surface.

As a consequence, the resulting fractional D3-brane solutions have naked singularities at small distance.

It is rather non-trivial to construct a fractional brane solution that satisfies both of the above criteria, namely at large distance and at small distance. The only examples we know so far are the deformed D3-brane in [10] and the two D2-branes that were obtained in [15] and in this paper.

For other branes, two cases can be considered. One possibility is that we can sacrifice the large-distance criterion that F_p carry non-vanishing flux, but still insist that the solution is regular everywhere. A large class of supersymmetric non-singular solutions of this type were constructed in [15, 18]. Since these solutions carry only the flux of F_n , but with no flux for F_p (or F_q), the resulting configurations are likely to describe perturbations of relevant operators in the dual field theories [20]. The second possibility is that one can instead sacrifice the short-distance criterion, and hope that the resulting naked singularity can be resolved by non-perturbative string effects, such as discussed in [30].

In the light of this discussion, we shall now examine the possibilities for deformations of various branes in $D = 10$ and $D = 11$.

The M2-brane has an 8-dimensional transverse space. In order to construct a fractional M2-brane, it would be necessary to have a Ricci-flat 8-manifold whose principal orbits (the 7-dimensional level-surfaces) have a non-vanishing 3-cycle. Unfortunately, such a configuration that is also supersymmetric does not seem to exist [20]. Although the large-distance criterion for the M2-brane cannot apparently be met, the short-distance criterion can often be satisfied. A large class of Ricci-flat metrics with L^2 -normalisable harmonic self-dual 4-forms were obtained in [15, 18], and these were used to construct various supersymmetric, non-singular deformed M2-branes.

The discussion for the NS-NS string in type IIA is the same as for the M2-brane, since it can be obtained by double-dimensional reduction. The situation for the NS-NS string in type IIB is rather different. It is S-dual to the D-string, which we shall discuss presently.

Fractional D-branes were discussed in [15, 20], and their large-distance asymptotic behaviours were classified in [20]. The largest fractional D-brane in supergravity is the D6-brane, making use of the Bianchi identity $dF_{(2)} = mF_{(3)}$, where m is the cosmological parameter in the massive type IIA supergravity. For D5-branes and D6-branes we have $p = 0$ and $p = 1$, and hence the solutions suffer from naked singularities at large distance, as we discussed previously. For the fractional D4-brane, the transverse space is 5-dimensional. Since there is no suitable non-trivial (irreducible) smooth Ricci-flat mani-

fold that admits covariantly-constant spinors, the resolution of the singularity lies beyond the level of supergravity. Resolutions for fractional D3-branes and D2-branes can be successfully implemented at the level of supergravity, and were discussed in detail in [10] (for D3-branes), and in [15] and this paper (for D2-branes).

3.1 Fractional D-strings

The fractional D-string has an 8-dimensional transverse space. Although there seem not to be examples where the principal orbits are 7-manifolds with non-trivial 3-cycles, thus precluding the construction of fractional M2-branes, there are examples with non-trivial 2-cycles. A D3-brane can wrap around such a 2-cycle, giving the possibility of a fractional D-string, provided that a corresponding harmonic 3-form exists on the 8-manifold. In order to avoid the singularity at short distance, the 8-metric should have the following short-distance behaviour:

$$ds_8^2 = d\rho^2 + \rho^2 d\Sigma_2^2 + d\Sigma_5^2. \quad (3.7)$$

This condition is necessary for the harmonic 3-form (or 5-form) to be square-integrable at short distance, which would then avoid the short-distance singularity.

Many Ricci-flat 8-manifolds are known, such as the hyper-Kähler Calabi metric, the Stenzel metric on T^*S^4 , the complex line bundles over \mathbb{CP}^3 or the 6-dimensional flag manifold $SU(3)/(U(1) \times U(1))$, and the Ricci-flat metric of $\text{Spin}(7)$ holonomy on the \mathbb{R}^4 bundle over S^4 . Other examples include various cases with principal orbits that are $U(1)$ bundles over products of Einstein-Kähler spaces, such as $\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$ or $\mathbb{CP}^1 \times \mathbb{CP}^2$. In these cases at most one of the \mathbb{CP}^m factors collapses to zero radius at short distance, implying that the manifold is of the form of a \mathbb{C}^k bundle over the remaining non-collapsed factors. The complex line bundles are particular examples, with $k = 1$.

The \mathbb{C}^k bundle metrics starting from $\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$ have the following short-distance structure:

$$ds_8 = d\rho^2 + \rho^2 d\nu^2 + (\rho^2 + \ell_1^2) d\Omega_1^2 + (\rho^2 + \ell_2^2) d\Omega_2^2 + (\rho^2 + \ell_3^2) d\Omega_3^2, \quad (3.8)$$

where $d\nu$ is equal to the sum of the volume forms on the three \mathbb{CP}^1 factors. If all three ℓ_i parameters are non-zero, then $k = 1$ and the manifold is a complex line bundle over $\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$. If in addition any two parameters ℓ_i are equal, the corresponding $\mathbb{CP}^1 \times \mathbb{CP}^1$ factors can be replaced by \mathbb{CP}^2 (see section 4 for a discussion of some global issues). If all three ℓ_i parameters are equal, the $\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$ can be replaced by any other Einstein-Kähler 6-manifold, such as \mathbb{CP}^3 or the 6-dimensional flag manifold $SU(3)/(U(1) \times U(1))$.

At most one of the ℓ_i parameters in (3.8) could instead be zero, giving a \mathbb{C}^2 bundle over $\mathbb{CP}^1 \times \mathbb{CP}^1$. Again, if the remaining non-zero parameters are then set equal, the $\mathbb{CP}^1 \times \mathbb{CP}^1$ could be replaced by \mathbb{CP}^2 . If parameters are set equal first, and a corresponding replacement by \mathbb{CP}^2 or \mathbb{CP}^3 made, one can set this parameter to zero and get a \mathbb{C}^3 bundle over \mathbb{CP}^1 , or \mathbb{C}^4 itself (flat space). (Again, see section 4 for a discussion of some global issues.)

The Ricci-flat metrics on these \mathbb{C}^k bundles over products of Einstein-Kähler spaces $M_1 \times M_2 \times \cdots \times M_N$ provide various resolutions of the cone metrics of the 7-spaces which are $U(1)$ bundles over $M_1 \times M_2 \times \cdots \times M_N$. These cone metrics were discussed in [20], as candidates for obtaining fractional D-strings. Their large-distance asymptotic behaviour is given by (3.2), with $n = 7$, $p = 5$, $q = 2$. Although the \mathbb{C}^k bundle metrics give smooth resolutions of the corresponding cone metrics, their short-distance behaviour is not, however, appropriate for allowing regular short-distance behaviour in the fractional D-string solutions. This is because, as can be seen from (3.8), they do not have non-collapsing 5-cycles at short distance. Thus they cannot admit appropriate harmonic 5-forms that could give rise to deformed D-strings with regular short-distance behaviour. It is, however, still of interest to study the short-distance singularity structure for the deformed fractional D-strings using these resolutions of the cone metrics.

Harmonic 5-forms for the metric (3.8) are given by

$$G_{(5)} = \nu \wedge (x_1 \Omega_1 \wedge \Omega_2 + x_2 \Omega_1 \wedge \Omega_3 + x_3 \Omega_2 \wedge \Omega_3), \quad (3.9)$$

where $x_1 + x_2 + x_3 = 0$. The magnitude of $G_{(5)}$ is then proportional to

$$|G_{(5)}|^2 \sim \frac{x_1^2 (\rho^2 + \ell_1^2)}{\rho (\rho^2 + \ell_2^2) (\rho^2 + \ell_3^2)} + \text{cyclic}, \quad (3.10)$$

and the determinant of the metric gives $\sqrt{g} \sim \rho (\rho^2 + \ell_1^2) (\rho^2 + \ell_2^2) (\rho^2 + \ell_3^2)$. Thus it follows that in the various distinct cases with vanishing or non-vanishing ℓ_i one has

$$\begin{aligned} \underline{\ell_i = (\ell_1, \ell_2, \ell_3)} : \quad & H \sim c_0 - m^2 (\log r)^2, \\ \underline{\ell_i = (\ell_1, \ell_2, 0)} : \quad & H \sim c_0 - \frac{m^2}{r^4}, \\ \underline{\ell_i = (\ell_1, 0, 0)} : \quad & H \sim c_0 - \frac{m^2}{r^8}, \\ \underline{\ell_i = (0, 0, 0)} : \quad & H \sim c_0 - \frac{m^2}{r^8}. \end{aligned} \quad (3.11)$$

(If more than one ℓ_i parameter vanishes, one would need at least to replace the associated S^2 factors by a \mathbb{CP}^2 or \mathbb{CP}^3 in order to avoid power-law curvature singularities at $r = 0$.) Presumably it could only be through non-perturbative string effects that any of the short-distance singularities in H could be resolved.

4 Further Ricci-flat metrics on \mathbb{C}^k bundles

In [18], a rather general class of non-compact Ricci flat manifolds was constructed. These are metrics of cohomogeneity one, whose principal orbits are $U(1)$ bundles over products of an arbitrary number N of compact Einstein-Kähler manifolds M_i . Typically, one might take each M_i factor to be a complex projective space, \mathbb{CP}^{m_i} . The Ricci-flat metrics constructed in [18] all had the feature that the radius of the $U(1)$ fibres and one of the M_i factors, say M_1 , collapsed to zero at small distance. The factor M_1 was therefore necessarily \mathbb{CP}^{m_1} , so that the collapsing submanifold was S^{2m_1+1} , and the metric could approach $\mathbb{R}^{2m_1+2} \times M_2 \times M_3 \times \cdots \times M_N$ locally at short distance. The manifolds were therefore \mathbb{C}^k bundles over $M_2 \times M_3 \times \cdots \times M_N$, with $k = m_1 + 1$. We refer back to [18] for a detailed discussion of the construction of these metrics.

In this section, we generalise the construction a little, to include the case where all the M_i factors remain uncollapsed at short distance, while the $U(1)$ fibres still shrink to zero. The structure of the manifold will therefore now be a complex line bundle over $M_1 \times M_2 \times \cdots \times M_N$. The metric ansatz, and the equations for Ricci-flatness, will be the same as in [18]. Thus we have

$$d\hat{s}^2 = e^{-2\gamma} dr^2 + e^{2\gamma} \sigma^2 + \sum_i e^{2\alpha_i} ds_i^2, \quad (4.1)$$

where ds_i^2 denotes the Einstein-Kähler metric on M_i , and

$$\sigma = d\psi + \sum_i A^i, \quad (4.2)$$

with $dA^i = p_i J^i$, where J^i is the Kähler form on M_i and p_i is an appropriately-chosen constant (see [18]). The functions α_i and γ depend only on r . (Note that we have made a different coordinate gauge choice for the radial variable, as compared with the one called r in [18].)

From results in [18], it follows that the metric will be Ricci flat if the following first-order equations, derivable from a superpotential, are satisfied:

$$\frac{d\alpha_i}{dr} = \frac{1}{2} p_i e^{-2\alpha_i}, \quad \frac{d\gamma}{dr} = k e^{-2\gamma} - \frac{1}{4} \sum_i n_i p_i e^{-2\alpha_i}, \quad (4.3)$$

where n_i is the real dimension of M_i . Note that k is a constant such that

$$\lambda_i = k p_i \quad (4.4)$$

for all i , where λ_i is the cosmological constant for ds_i^2 [18].

We now solve the first-order equations (4.3). For α_i , we find

$$e^{2\alpha_i} = p_i (r + \ell_i^2), \quad (4.5)$$

where the ℓ_i are arbitrary constants of integration. The equation for γ can then be solved, to give

$$e^{2\gamma} = 2k \prod_i (r + \ell_i^2)^{-n_i/2} \int_0^r dy \prod_j (y + \ell_j^2)^{n_j/2}. \quad (4.6)$$

The integration is elementary, giving an expression for $e^{2\gamma}$ as a rational function of r for any given choice of the integers n_i , but the general expression for arbitrary dimensions n_i requires the use of hypergeometric functions. The Ricci-flat metrics are therefore given by

$$d\hat{s}^2 = e^{-2\gamma} dr^2 + e^{2\gamma} \sigma^2 + \sum_i p_i (r + \ell_i^2) ds_i^2, \quad (4.7)$$

with γ given by (4.6).

The radial coordinate r always runs from $r = 0$ to $r = \infty$. One easily sees that as r tends to infinity, the metric (4.1) tends to

$$d\hat{s}^2 = d\rho^2 + \rho^2 d\bar{s}^2, \quad (4.8)$$

where

$$d\bar{s}^2 = \frac{4k^2}{D^2} \sigma^2 + \frac{k}{D} \sum_i p_i ds_i^2, \quad (4.9)$$

$D = 2 + \sum_i n_i$ denotes the total dimension of $d\hat{s}^2$, and we have defined a new radial coordinate ρ given by $r = k \rho^2 / D$. It is easy to verify that $d\bar{s}^2$ is an Einstein metric on the $U(1)$ bundle over $\prod_i M_i$, with $\bar{R}_{ab} = (D - 2) \bar{g}_{ab}$. Thus (4.1) approaches the cone metric at large r .

It is appropriate at this juncture to discuss certain global and topological issues. These concern the periodicity of the $U(1)$ fibre coordinate ψ , and the conditions under which one obtains a regular manifold. The discussion divides into two parts. Firstly, we must ensure that the principal orbits themselves are regular. These are the level surfaces with $r > 0$, for which all of the metric functions $e^{2\gamma}$ and $e^{2\alpha_i}$ are positive. The level surfaces are $U(1)$ bundles over the product $\prod_i M_i$ of Einstein-Kähler base manifolds. Having established when the principal orbits are regular, there is a remaining global question of whether the metric behaves appropriately as the orbits degenerate at $r = 0$, so as to give a smooth manifold.

Considering first the principal orbits, the period $\Delta\psi$ of the $U(1)$ fibre coordinate ψ must be compatible with the integrals of the curvature of the fibre over all 2-cycles in the base

space $\prod_i M_i$. Specifically, using (4.2), and recalling that $dA^i = p_i J^i$, this means that we must have

$$\Delta\psi = \frac{p_i}{q_{ij}} \int_{S_{ij}} J^i, \quad \text{for all the } i, j, \quad (4.10)$$

where S_{ij} denotes the j 'th 2-cycle in the manifold M_i , and q_{ij} are integers. The manifold of the $U(1)$ bundle over $\prod_i M_i$ will be simply connected if these integers are chosen to be as small possible; let us denote this manifold by \mathcal{M} . One can also have non-simply-connected smooth manifolds in which $\Delta\psi$ is taken to be this maximum allowed period divided by any integer s ; these will be the manifolds \mathcal{M}/\mathbb{Z}_s .

Since each M_i is Einstein-Kähler, with Kähler form J^i and cosmological constant λ_i , we can write its Ricci form as $\mathcal{R}_i = \lambda_i J^i$. The Ricci form is 2π times the first Chern class, and so

$$\frac{1}{2\pi} \int_{S_{ij}} \mathcal{R}_i = h_{ij}, \quad (4.11)$$

where h_{ij} are a set of integers, labelled by j , characteristic of each manifold M_i , and determined purely by its topology. Bearing in mind the relation (4.4), it is then easy to see that the maximum allowed period for ψ , compatible with all the integrals over 2-cycles, will be⁵

$$(\Delta\psi)_{\max} = \frac{2\pi}{k} \gcd(h_{ij}), \quad (4.12)$$

where $\gcd(h_{ij})$ denotes the greatest common divisor of all the (non-vanishing) integers h_{ij} . With this period, the principal orbits \mathcal{M} will be simply-connected; one can instead take the period to be $(\Delta\psi)_{\max}/s$ for any integer s , giving non-simply-connected principal orbits \mathcal{M}/\mathbb{Z}_s .

The situation becomes simple if all the factors in the base space are taken to be complex projective spaces, $M_i = \mathbb{CP}^{m_i}$, since then there is only one 2-cycle in each factor M_i , and furthermore we know that the associated integer h_{i1} is given by $h_{i1} = m_i + 1$. In fact, it is convenient in this case to make convention choices so that we have

$$\lambda_i = 2m_i + 2, \quad p_i = m_i + 1, \quad k = 2. \quad (4.13)$$

We therefore have that the maximum period is given by

$$(\Delta\psi)_{\max} = \pi \gcd(m_1 + 1, m_2 + 1, \dots, m_N + 1). \quad (4.14)$$

⁵Note that the method we have used in order to obtain this topological information involves the use of the metric on the base manifold. We have done this for convenience, since it provides a simple way to obtain the results, but it is worth emphasising that it is possible instead to obtain the same results without needing to make use of the metric on $\prod_i M_i$ at all.

The discussion above was concerned with the conditions for regularity of the principal orbits themselves. There are further regularity considerations involving the structure of the metric near to $r = 0$. The discussion bifurcates, depending on whether all the ℓ_i are non-vanishing or not. In fact it is easy to see that regularity at $r = 0$ implies that at most one of the ℓ_i can be zero.

Consider first the case where all the ℓ_i are non-zero. Introducing a new radial coordinate ρ defined by $r = k \rho^2/2$, we find that near $\rho = 0$ the metric (4.1) approaches

$$d\hat{s}^2 = d\rho^2 + k^2 \rho^2 \sigma^2 + \sum_i p_i \ell_i^2 ds_i^2. \quad (4.15)$$

This will be regular at $\rho = 0$, approaching $\mathbb{R}^2 \times \prod_i M_i$ locally, provided that the $U(1)$ fibre coordinate ψ has period $2\pi/k$. From (4.12), we see that this is always possible. Generically, the set of integers h_{ij} will have no common divisor other than 1, and so the case with simply-connected principal orbits will be the regular one. If instead there is a greatest common divisor s , then the principal orbits will need to be factored by \mathbb{Z}_s to get the regular total Ricci-flat manifold.

For the special cases where there is a single Einstein-Kähler base-space factor M_1 , the Ricci-flat metrics of this type fall into the class constructed in [31, 32]. An example would be the line bundle over the Einstein-Kähler metric on $S^2 \times S^2$. A new special case with two factors, each of which is S^2 , but with parameters ℓ_1 and ℓ_2 now unequal, was recently obtained in [21] (in a different system of coordinates).

Consider now the case where one of the constants ℓ_i is zero; this class of metrics was discussed in [18]. Without loss of generality, we may assume that $\ell_1 = 0$. In terms of a new radial variable ρ , defined by $r = k \rho^2/(n_1 + 2)$, we find that near $\rho = 0$ the metric (4.1) approaches

$$d\hat{s}^2 = d\rho^2 + \frac{4k^2}{(n_1 + 2)^2} \rho^2 \sigma^2 + \frac{p_1 k}{n_1 + 2} \rho^2 ds_1^2 + \sum_{i \geq 2} p_i \ell_i^2 ds_i^2. \quad (4.16)$$

For this to be regular at $r = 0$, it is necessary that the terms involving σ^2 and ds_1^2 give rise to a sphere. A *sine qua non* for this to happen is that the manifold M_1 must be the complex projective space \mathbb{CP}^{m_1} with its Fubini-Study metric. However, it is also necessary that the maximum allowed period for ψ , determined by (4.12), be equal to the maximum period that ψ would be allowed if there were no additional factors M_2, M_3, \dots in the base space, since only then will we get S^{2m_1+1} itself, rather than a factoring of it. In other words, regularity at $r = 0$ requires that

$$m_1 + 1 = \gcd(m_1 + 1, h_{2j}, h_{3j}, \dots, h_{Nj}). \quad (4.17)$$

For example, in the case where all the base-space factors are complex projective spaces, $M_i = \mathbb{CP}^{m_i}$, we must have

$$m_1 + 1 = \gcd(m_1 + 1, m_2 + 1, m_3 + 1, \dots, m_N + 1). \quad (4.18)$$

If, for instance, we have $M_1 = \mathbb{CP}^1 = S^2$, then we can have $\mathbb{CP}^1, \mathbb{CP}^3, \mathbb{CP}^5, \mathbb{CP}^7$, *etc.*, for the other factors, but not $\mathbb{CP}^2, \mathbb{CP}^4$, *etc.* If we have $M_1 = \mathbb{CP}^2$, then we can have $\mathbb{CP}^2, \mathbb{CP}^5, \mathbb{CP}^8, \mathbb{CP}^{11}$, *etc.*, for the other factors, but not $\mathbb{CP}^1, \mathbb{CP}^3, \mathbb{CP}^4, \mathbb{CP}^6, \mathbb{CP}^7$, *etc.* Note that $M_i = \mathbb{CP}^m$, with all factors the same, is allowed for any m .

The ℓ_i parameters are moduli that parameterise the radii of cycles in the total manifold. Setting one of the ℓ_i to zero, which is a smooth transformation in the modulus space, however corresponds to a topology-changing transformation in the manifold of the \mathbb{C}^k bundle over $\prod_i M_i$.

5 Conclusions

The principal focus of this paper was a study of the resolution of brane configurations with additional fluxes that have non-vanishing integrals at infinity (*i.e.* non-vanishing charges). These configurations, which also have an interpretation in terms of the wrapping of branes around the associated cycles, are referred to as fractional branes. The transverse space is a complete non-compact Ricci-flat manifold.

The examples we studied in greatest detail were associated with fractional D2-branes. The space transverse to a D2-brane is a seven-dimensional Ricci-flat manifold. In order to describe coincident D2-branes rather than an array, this seven-manifold should be asymptotically conical. When the level surfaces (or principal orbits) have non-trivial homology, the possibility arises that other kinds of brane can wrap around the cycles. The structure of the transgression terms, and supersymmetry, imply that only D4-branes or NS-NS 5-branes can wrap; around 2-cycles or 3-cycles respectively. The wrapped D4-branes are referred to as fractional D2-branes, and they are supported by magnetic charges carried by the 4-form field strength of the type IIA theory. This contrasts with the electric charges carried by the usual D2-branes. The wrapped NS-NS 5-branes can be viewed as fractional NS-NS 2-branes, since they are supported by magnetic charges carried by the NS-NS 3-form.

In this paper, we constructed completely regular fractional D2-brane solutions, using for the transverse space a Ricci-flat 7-manifold of G_2 holonomy that was obtained in [27, 28]. This manifold has level surfaces that are topologically \mathbb{CP}^3 (arising as an S^2 bundle over S^4), and can be described as the bundle of self-dual 2-forms over S^4 . (A second

possibility is to take the analogous 7-manifold in which the S^4 is replaced by \mathbb{CP}^2 , leading to similar conclusions.) In order to obtain such a resolved fractional D2-brane solution we first constructed an L^2 -normalisable harmonic 3-form. At large distance, in the decoupling limit, the solution has the same asymptotic form as that of the regular D2-brane with a flat transverse space, however, its charge is determined by the charge of the fractional D2-brane. This result indicates that the asymptotic (ultraviolet) limit of the dual field theory is the same as for the regular D2-brane. However the SYM factor is governed by the fractional brane charge.

We then discussed the example of a fractional NS-NS 2-brane, for which a regular solution was obtained previously in [15]. This uses another Ricci-flat manifold of G_2 holonomy that was found in [27, 28], for which the level surfaces are topologically $S^3 \times S^3$. Note that in this case while the harmonic 3-form $G_{(3)}$ that supports the NS-NS 5-brane flux is square-integrable at small distance, it is not L^2 -normalisable, owing to a linear divergence of the integral of $|G_{(3)}|^2$ at large distance. This means that asymptotically, the overall “charge” of this resolved solution with NS-NS 5-branes wrapped over 3-cycles grows linearly with the distance. Note that, as in the previous example with D4-branes wrapped over 2-cycles, the constant part of the charge is determined by the fractional brane charge (NS-NS 2-brane charge in this latter case). The linear dependence of the charge on the distance indicates that in the dual field theory the asymptotic renormalisation of the difference of gauge couplings may grow linearly with the energy scale.

We also showed that both the above resolved D2-brane solutions are supersymmetric. Specifically, the fraction of preserved supersymmetry in each case is the same as the fraction that is preserved by the usual D2-brane, with no fractionally charged branes, after taking into account the supersymmetry reduction already implied by the replacement of the flat transverse 7-space by the Ricci-flat manifold of G_2 holonomy. Thus the supersymmetry on the world volume of the D2-brane will be $\mathcal{N} = 1$, as opposed to the usual $\mathcal{N} = 8$ of a normal D2-brane with a flat transverse space. This result implies that the supergravity solutions are dual to $\mathcal{N} = 1$ three-dimensional gauge theories. Note that the two types of resolved D2-brane solutions, *i.e.* those with fractional D2-branes and those with fractional NS-NS 2-branes, have qualitatively the same small-distance behaviour, thus indicating the universality of the dual $\mathcal{N} = 1$ field theory in the infrared regime. On the other hand the ultraviolet behaviour of these two types of solutions is qualitatively different, and of course neither of them has any restoration of conformal symmetry.⁶ We have also analysed

⁶ We should contrast these results with those for the resolved M2-branes in [15, 18], for which the conformal symmetry of $\text{AdS}_4 \times M_7$ is restored asymptotically, in the decoupling limit. Since the resolved

the spectra of the minimally-coupled scalar equations in both the resolved D2-brane backgrounds. The spectra are discrete and qualitatively similar, indicating confinement in the infrared regime of the dual three-dimensional field theories.

Our subsequent additional analysis shows that regular supersymmetric fractional brane solutions are rather rare in supergravities. The only other example known to date is the deformed fractional D3-brane obtained in [10]. We examined possible resolutions for other fractional branes, and derived necessary conditions for regularity, but these were not fulfilled in any of these other examples. It is quite possible that the resolutions of these other examples involve a resort to non-perturbative effects in string theory, such as are discussed in [30].

For some of the other examples that we studied, we took as their starting point singular deformations that employ the cone metrics discussed in [20]. In these examples the level surfaces in the cone are $U(1)$ bundles over products of complex projective spaces. In fact a large class of complete Ricci-flat manifolds that smooth out the singularity at the apex of the cone were discussed in [18], and in this paper we used these in order to try to obtain resolutions of the cone deformations discussed in [20]. These did not yield regular resolutions for fractional D-strings, however, owing to the fact that none of them has non-collapsing 5-cycles at short distance. As a by-product we also extended the class of Ricci-flat metrics constructed in [18].

Various directions for further investigation remain. Firstly, it would be very interesting to investigate further the dual field theories for the resolved D2-branes with fractional branes. Secondly, it would be nice to establish whether further examples of regular supersymmetric fractional branes exist at the level of supergravity. Furthermore, for cases where this is not possible, one would like to gain a deeper understanding of how a non-perturbative string mechanism that could resolve the problems might operate.

Finally, there also exist different types of regular supersymmetric resolved brane solutions, that again make use of the transgression terms, and harmonic forms on the transverse Ricci-flat manifolds, but which have no charges associated with the additional fluxes and thus do not correspond to fractional branes. Many explicit examples, such as resolved M2-branes, heterotic 5-brane and dyonic strings, were constructed in [15, 18] and they all restore the asymptotic conformal symmetry in the decoupling limit. Dual field theories were conjectured to correspond to the Higgs phase [18], and evidence was presented [20] M2-brane examples have no charge associated with the additional fluxes, there are no fractional branes there and the dual field theory interpretation is that of a three-dimensional CFT perturbed by relevant operators [20].

which shows that in the resolved M2-brane examples the dual CFT is perturbed by relevant operators. It would be of interest to investigate whether further such solutions exist, and to gain a better understanding of their rôle in the framework of the AdS/CFT correspondence.

Acknowledgements

We should like to thank Nigel Hitchin, Igor Klebanov and Edward Witten for discussions. M.C. is supported in part by DOE grant DE-FG02-95ER40893 and NATO grant 976951; H.L. is supported in full by DOE grant DE-FG02-95ER40899; C.N.P. is supported in part by DOE DE-FG03-95ER40917. The work of M.C., G.W.G. and C.N.P. was supported in part by the programme *Supergravity, Superstrings and M-theory* of the Centre Émile Borel of the Institut Henri Poincaré, Paris (UMS 839-CNRS/UPMC).

References

- [1] J. Maldacena, *The large N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. **2** (1998) 231, hep-th/9711200.
- [2] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, Phys. Lett. **B428** (1998) 105, hep-th/9802109.
- [3] E. Witten, *Anti-de Sitter space and holography*, Adv. Theor. Math. Phys. **2** (1998) 253, hep-th/980215.
- [4] I.R. Klebanov and A.A. Tseytlin, *Gravity duals of supersymmetric $SU(N) \times SU(N+m)$ gauge theories*, Nucl. Phys. **B578**, 123 (2000) [hep-th/0002159].
- [5] P. Candelas and X.C. de la Ossa, *Comments on conifolds*, Nucl. Phys. **B342**, 246 (1990).
- [6] E.G. Gimon and J. Polchinski, *Consistency conditions for orientifolds and D manifolds*, Phys. Rev. **D54**, 1667 (1996), [hep-th/9601038].
- [7] M.R. Douglas, *Enhanced gauge symmetry in M (atrix) theory*, JHEP **007**, 004 (1997), [hep-th/9612126].
- [8] S.S. Gubser and I.R. Klebanov, *Baryons and domain walls in an $N = 1$ superconformal gauge theory*, Phys. Rev. **D58**, 125025 (1998), [hep-th/9808075].

- [9] I.R. Klebanov and N. Nekrasov, *Gravity duals of fractional branes and logarithmic RG flow*, Nucl. Phys. **B574**, 263 (2000), [hep-th/9911096].
- [10] I.R. Klebanov and M.J. Strassler, *Supergravity and a confining gauge theory: duality cascades and χ SB-resolution of naked singularities*, JHEP **0008**, 052 (2000), [hep-th/0007191].
- [11] M.B. Stenzel, *Ricci-flat metrics on the complexification of a compact rank one symmetric space*, Manuscripta Mathematica, **80**, 151 (1993).
- [12] M. Graña and J. Polchinski, *Supersymmetric three-form flux perturbations on AdS_5* , hep-th/0009211.
- [13] S.S. Gubser, *Supersymmetry and F-theory realization of the deformed conifold with three-form flux*, hep-th/0010010.
- [14] L.A. Pando Zayas and A.A. Tseytlin, *3-branes on a resolved conifold*, hep-th/0010088.
- [15] M. Cvetič, H. Lü and C.N. Pope, *Brane resolution through transgression*, hep-th/0011023.
- [16] F. Bigazzi, L. Giradello and A. Zaffaroni, *A note on regular type 0 solutions and confining gauge theories*, hep-th/0011041.
- [17] M. Bertolini, P. Di Vecchia, M. Frau, A. Lerda, R. Marotta and I. Pesando, *Fractional D-branes and their gauge duals*, hep-th/0011077.
- [18] M. Cvetič, G.W. Gibbons, H. Lü and C.N. Pope, *Ricci-flat metrics, harmonic forms and brane resolutions*, hep-th/0012011.
- [19] O. Aharony, *A note on the holographic interpretation of string theory backgrounds with varying flux*, hep-th/0101013.
- [20] C.P. Herzog and I.R. Klebanov, *Gravity duals of fractional branes in various dimensions*, hep-th/0101020.
- [21] L.A. Pando Zayas and A.A. Tseytlin, *3-branes on spaces with $\mathbb{R} \times S^2 \times S^3$ topology*, hep-th/0101043.
- [22] M.J. Duff, J.M. Evans, R.R. Khuri, J.X. Lu and R. Minasian, *The octonionic membrane*, Phys. Lett. **B412**, 281 (1997) [hep-th/9706124].

- [23] S.W. Hawking and M.M. Taylor-Robinson, *Bulk charges in eleven dimensions*, Phys. Rev. **D58** 025006 (1998) [hep-th/9711042].
- [24] B.R. Greene, K. Schalm and G. Shiu, *Warped compactifications in M and F theory*, Nucl. Phys. **B584**, 480 (2000) [hep-th/0004103].
- [25] K. Becker and M. Becker, *Compactifying M-theory to four dimensions*, JHEP **0011**, 029 (2000) [hep-th/0010282].
- [26] K. Becker, *A note on compactifications on Spin(7)-holonomy manifolds*, hep-th/0011114.
- [27] R.L. Bryant and S. Salamon, *On the construction of some complete metrics with exceptional holonomy*, Duke Math. J. **58**, 829 (1989).
- [28] G.W. Gibbons, D.N. Page and C.N. Pope, *Einstein metrics on S^3 , \mathbb{R}^3 and \mathbb{R}^4 bundles*, Commun. Math. Phys. **127**, 529 (1990).
- [29] B.S. Acharya, J.M. Figueroa-O'Farrill, C.M. Hull and B. Spence, *Branes at conical singularities and holography*, Adv. Theor. Math. Phys. **2**, 1249 (1999) [hep-th/9808014].
- [30] C. V. Johnson, A. W. Peet and J. Polchinski, *Gauge theory and the excision of repulson singularities*, Phys. Rev. D **61**, 086001 (2000) [hep-th/9911161].
- [31] L. Berard-Bergery, *Quelques exemples de varietes riemanniennes completes non compactes a courbure de Ricci positive*, C.R. Acad. Sci., Paris, Ser. I302, 159 (1986).
- [32] D.N. Page and C.N. Pope, *Inhomogeneous Einstein metrics on complex line bundles*, Class. Quantum Grav. **4**, 213 (1987).